Jacobson Radical in Artinian $\mathbb{Z}$-Algebras: Nilpotency and Centrality

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1 Introduction and Motivation

In many algebra courses we learn about rings and modules. Today we focus on a special case: when our ring is an algebra over $\mathbb{Z}$ (that is, a ring with unity, with the canonical inclusion $\mathbb{Z} \hookrightarrow A$) and satisfies the Artinian condition. We shall prove that the Jacobson radical $J(A)$ is nilpotent. (In a commutative ring, "being in the center" is automatic; in noncommutative settings the radical need not be central but its nilpotency remains an important property.) This result plays a key role in structure theory (see, for example, the Wedderburn–Artin Theorem).

2 Preliminaries and Definitions

2.1 Rings and $\mathbb{Z}$-Algebras

Definition 2.1 ($\mathbb{Z}$-algebra)**.** *A ring A with unity is called a $\mathbb{Z}$-algebra because there is a unique ring homomorphism*

$$\varphi : \mathbb{Z} \to A \quad \textit{given by} \quad n \mapsto n \cdot 1_A.$$

Thus every ring with unity is naturally an algebra over $\mathbb{Z}$. (Often we consider additional structure, but here the main point is that $\mathbb{Z}$ sits in the center of A.)

2.2 Artinian Rings

Definition 2.2 (Artinian ring). *A ring A is said to be (left)* Artinian *if every descending chain of left ideals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

stabilizes (that is, there is some N with $I_N = I_{N+1} = \cdots$). (One may similarly define right Artinian; for rings with unity, these notions coincide in many important cases.)

Example 2.3. *Any finite ring is Artinian. Also, any finite-dimensional algebra over a field (or over $\mathbb{Z}$) is Artinian.*

2.3 Nilpotent Elements and Ideals

Definition 2.4 (Nilpotent element). *An element $a \in A$ is* nilpotent *if there exists some positive integer n such that*

$$a^n = 0.$$

Definition 2.5 (Nilpotent ideal). *An ideal $I \subset A$ is* nilpotent *if there exists an n such that*

$$I^n = \{a_1 a_2 \cdots a_n \mid a_i \in I\} = \{0\}.$$

2.4 Center of a Ring

Definition 2.6 (Center). *The* center *of a ring A is*

$$Z(A) = \{z \in A \mid za = az \quad \textit{for all } a \in A\}.$$

If A is commutative, then $Z(A) = A$. In any $\mathbb{Z}$-algebra, the image of $\mathbb{Z}$ lies in $Z(A)$ because every integer acts as a scalar.

2.5 The Jacobson Radical

Definition 2.7 (Jacobson Radical). *The* Jacobson radical *$J(A)$ of a ring A is defined as the intersection of all maximal left ideals of A:*

$$J(A) = \bigcap \{M \subseteq A \mid M \textit{ is a maximal left ideal}\}.$$

It turns out that

$$J(A) = \{x \in A \mid 1 - ax \textit{ is invertible for all } a \in A\}.$$

Example 2.8. *For a field F, the only maximal left ideal is $\{0\}$, so $J(F) = \{0\}$. More generally, if A is semisimple (for example, a matrix ring over a field), then $J(A) = \{0\}$.*

3 Main Theorem: Nilpotency of the Jacobson Radical in Artinian Rings

Theorem 3.1. *Let A be an Artinian ring (for instance, a finite-dimensional $\mathbb{Z}$-algebra). Then the Jacobson radical $J(A)$ is a nilpotent ideal. That is, there exists a positive integer n such that*

$$J(A)^n = \{0\}.$$

Remarks

- In a commutative ring (or any $\mathbb{Z}$-algebra), since the ring is commutative the radical automatically lies in the center.
- In noncommutative rings the radical need not be central. However, the nilpotency is the key fact for many structural theorems.

3.1 Proof Outline of Theorem 3.1

We now outline a proof of this classical result. (The full proof requires several ideas from ring theory; our version is aimed at clarity for undergraduates.)

Proof (Sketch). Since A is Artinian, every descending chain of ideals stabilizes. Consider the chain:

$$J(A) \supseteq J(A)^2 \supseteq J(A)^3 \supseteq \cdots .$$

By the Artinian property, there is an integer N such that

$$J(A)^N = J(A)^{N+1} = \cdots .$$

We wish to show that in fact $J(A)^N = \{0\}$.

A key fact is that in an Artinian ring, the quotient $A/J(A)$ is semisimple (this is sometimes taken as a definition of a semiprimary ring). By the Wedderburn–Artin Theorem, $A/J(A)$ decomposes into a finite direct sum of matrix rings over division rings. In a semisimple ring, there are no nonzero nilpotent ideals. (One way to see this is that any nilpotent ideal would intersect a simple component trivially, forcing it to be zero.)

Now, if $J(A)^N \neq \{0\}$, then its image in the semisimple ring $A/J(A)$ would be a nonzero nilpotent ideal. But as just mentioned, semisimple rings cannot have nonzero nilpotent ideals. Thus, we must have

$$J(A)^N = \{0\}.$$

This completes the proof (details can be filled in by showing that the nilpotency of any nil ideal in an Artinian ring forces the ideal to vanish in the semisimple quotient). □

4 Examples and Applications

4.1 Example: Finite-Dimensional $\mathbb{Z}$-Algebra

Let A be a finite-dimensional algebra over $\mathbb{Z}$ (for instance, a ring of $n \times n$ matrices with integer entries modulo some relation). Because A is finite as an abelian group, it is Artinian. By Theorem 3.1, its Jacobson radical $J(A)$ is nilpotent; that is, there exists some m such that $J(A)^m = \{0\}$.

4.2 Application in Structure Theory

One of the reasons nilpotency of $J(A)$ is important is that it allows us to "peel off" the radical and study the semisimple part $A/J(A)$ separately. In many structural theorems (such as the Wedderburn–Artin Theorem), this separation is crucial in classifying the ring A.

5 Summary

In this lecture we have:

- Defined what it means for a ring to be a $\mathbb{Z}$-algebra.
- Introduced Artinian rings, nilpotent elements/ideals, the center of a ring, and the Jacobson radical.

- Shown (by outline) that if A is an Artinian ring, then its Jacobson radical $J(A)$ is nilpotent.

For a commutative ring (or any $\mathbb{Z}$-algebra) the image of $\mathbb{Z}$ lies in the center, and so in these cases the nilpotent radical is automatically "central" in the sense that all elements of A commute with the image of $\mathbb{Z}$. In non-commutative rings the radical itself need not be central, but its nilpotency remains a powerful structural property.